TORSIONAL VIBRATIONS WITH MULTIPLE MODES

MULTIPLE INERTIA SYSTEM

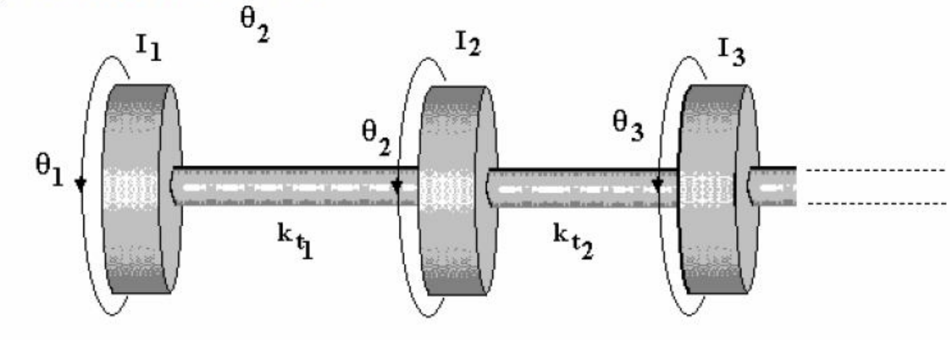

Figure 1

If we have several discs on a shaft as shown, there are several possible nodes and natural frequencies. There is more than one mode of oscillation possible. A method of solving this system is due to Holzer. The reasoning goes like this.

Let disc 1 twist relative to disc 2. The toque balance gives $I_1 \alpha_1 + k_{t1}(\theta_1 - \theta_2) = 0$

Let disc 2 twist relative to discs 1 and 3. The toque balance gives $I_2 \alpha_2 + k_{t1}(\theta_2 - \theta_1) + k_{t2}(\theta_2 - \theta_3) = 0$

Let disc 3 twist relative to disc 2. The toque balance gives $I_3 \alpha_3 + k_{t2}(\theta_3 - \theta_2) = 0$

For simple harmonic motion we may substitute $\omega^2\theta = -\alpha$ into each equation and rearrange them to give

$I_1 \omega_1^2 \theta_1 = k_{t1}(\theta_1 - \theta_2)$

$I_2 \omega_2^2 \theta_2 = k_{t1}(\theta_2 - \theta_1) + k_{t2}(\theta_2 - \theta_3)$

$I_3 \omega_3^2 \theta_3 = k_{t2}(\theta_3 - \theta_2)$

If we add all three equation we find $I_1 \omega_1^2 \theta_1 + I_2 \omega_2^2 \theta_2 + I_3 \omega_3^2 \theta_3 = 0$

For any number of discs this may be generalised as $\Sigma I \omega^2 \theta = 0$

Holzer's method of solution proposes that we assume any value of ω and make $\theta_1 = 1$ and calculate all the other deflections.

The deflection of disc 2 may be found by rearranging $I_1 \omega_1^2 \theta_1 = k_{t1}(\theta_1 - \theta_2)$ to give $\theta_2 = \theta_1 - \frac{\omega^2}{k_{t1}} I_1\theta_1$

The deflection of disc 3 may be found by rearranging $I_2 \omega_2^2 \theta_2 = k_{t1}(\theta_2 - \theta_1) + k_{t2}(\theta_2 - \theta_3)$

To do this substitute $\theta_1 = \theta_2 + \frac{\omega^2}{k_{t1}} I_1\theta_1$ and we have

$$\omega^2 I_2\theta_2 = k_{t1}\left(\theta_2 - \theta_2 - \frac{\omega^2}{k_{t1}} I_1\theta_1\right) + k_{t2}(\theta_2 - \theta_3)$$

$$\omega^2 I_2\theta_2 = -\omega^2 I_1\theta_1 + k_{t2}(\theta_2 - \theta_3)$$

$$k_{t2}\theta_3 = k_{t2}\theta_2 - \omega^2 I_2\theta_2 - \omega^2 I_1\theta_1$$

$$k_{t2}\theta_3 = k_{t2}\theta_2 - \omega^2 (I_1\theta_1 + I_2\theta_2)$$

$$\theta_3 = \theta_2 - \frac{\omega^2}{k_{t2}}(I_1\theta_1 + I_2\theta_2)$$

If this was continued the pattern for any number of discs would be as follows.

$$\theta_2 = \theta_1 - \frac{\omega^2}{k_{t1}} I_1\theta_1$$

$$\theta_3 = \theta_2 - \frac{\omega^2}{k_{t2}} (I_1\theta_1 + I_2\theta_2)$$

$$\theta_4 = \theta_3 - \frac{\omega^2}{k_{t3}} (I_1\theta_1 + I_2\theta_2 + I_3\theta_3)$$

And so on for as many as exist.

Next we consider the torque produced by the twisting. $T = I\,\alpha$ and $\alpha = \omega^2\theta$ so $T = \omega^2 I\,\theta$.

The torque to deflect disc 1 by θ_1 is $\omega^2 I_1\,\theta_1$

The torque to deflect disc 2 by θ_2 is $\omega^2 I_2\,\theta_2$

The torque to deflect disc 3 by θ_3 is $\omega^2 I_3\,\theta_3$

And so on for as many shaft section that exist.

Hence

$T_1 = \omega^2 I_1\,\theta_1$

$T_2 = T_1 + \omega^2 I_2\,\theta_2$

$T_3 = T_2 + \omega^2 I_3\,\theta_3$

And so on for as many shaft section that exist.

Since we must satisfy $\Sigma I\,\omega^2\,\theta = 0$ then the last T must be zero when the oscillation is free. The problem is to find the values of ω that make this so and these are the natural frequencies of the system.

If a computer programme is used, it is relatively simple to evaluate the displacements and the torques for all values of ω. Before we look at difficult problems let's consider the case of only two rotors.

TWO INERTIA SYSTEM

Consider a shaft with torsional stiffness k_t connecting two inertias I_1 andI_2. If the shaft is free to rotate the torsional oscillation will take the form of both ends twisting but some point in between will not be twisting. This is a node. The shaft must of course be supported in at least two bearings.

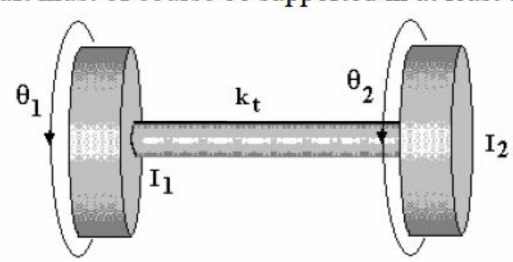

Figure 2

The natural frequency can be derived from the previous work. For two rotors, $T_2 = 0$

$$\theta_2 = \theta_1 - \frac{\omega^2 I_1\theta_1}{k_{t1}} \quad T_1 = \omega^2 I_1\theta_1 \quad T_2 = T_1 + \omega^2 I_2\theta_2 = 0 \quad \text{substitute for } \theta_1$$

$$T_2 = 0 = T_1 + \omega^2 I_2\theta_2 = \omega^2 I_1\theta_1 + \omega^2 I_2\theta_2 \ \text{ substitute for } \theta_2$$

$$0 = \omega^2 I_1\theta_1 + \omega^2 I_2\left(\theta_1 - \frac{\omega^2 I_1\theta_1}{k_{t1}}\right) \ \text{ simplify and rearrange to get}$$

$$\omega_n^2 = k_t\frac{I_1 + I_2}{I_1 I_2}$$

The node will be somewhere between the two rotors

WORKED EXAMPLE No.1

A shaft free to rotate carries a flywheel with $I_1 = 2$ kg m^2 at one end and $I_2 = 4$ kg m^2 at the other. The shaft connecting them has a stiffness of 4 MN m/rad. Calculate the natural frequency and the position of the node.

SOLUTION

$$\omega_n^2 = k_t \frac{I_1 + I_2}{I_1 I_2} = 4 \times 10^6 \frac{2+4}{2 \times 4} = 3 \times 10^6 \quad \omega_n = 1732 \text{ rad/s} \quad f_n = 275.7 \text{ Hz}$$

If we regard the node as a fixed point each rotor will have the same natural frequency about that point. For a single rotor system $\omega_n^2 = k_t/I$.
For the first rotor $\omega_n^2 = 3 \times 10^6 = k_{t1}/2$ $k_{t1} = 6 \times 10^6$
For the other rotor $\omega_n^2 = 3 \times 10^6 = k_{t2}/4$ $k_{t2} = 12 \times 10^6$
The difference in stiffness is due to the difference in length of the shaft. $k_t = GJ/L$ and GJ is the same for both sections.

$k_{t1}/k_{t2} = L_2/L_1 = 6/12$ $L_2 = L_1/2$ and $L_1 + L_2 = L$ $L_2 = (L - L_2)/2$
$2L_2 = L - L_2$
$3L_2 = L$
$L_2 = L/3$ $L_1 = 2L/3$ so the node is L/3 from the right.

This may be found another way. Let $\theta_1 = 1$

$$\theta_2 = \theta_1 - \frac{\omega^2 I_1 \theta_1}{k_{t1}} = 1 - \frac{3 \times 10^6 \times 2 \times 1}{4 \times 10^6} = 1 - 1.5 = -0.5$$

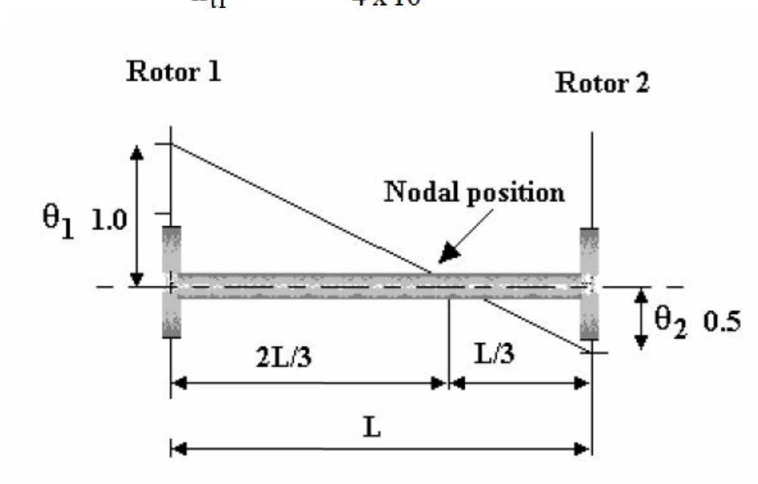

Figure 3

WORKED EXAMPLE No.2

A shaft has three inertias on it of 2, 4 and 2 kg m^2 respectively viewed from left to right. The shaft connecting the first two has a stiffness of 3 MN m/radian and the shaft connecting the last two has a stiffness of 2 MN m/radian. The system is supported in bearings at both ends. Ignore the inertia of the shafts and find the natural frequencies of the system.

SOLUTION

$$\theta_1 = 1 \qquad \theta_2 = \theta_1 - \frac{\omega^2}{k_{t1}} I_1\theta_1 = 1 - \frac{2\omega^2}{3 \times 10^6} \qquad \theta_3 = \theta_2 - \frac{\omega^2}{k_{t2}}(I_1\theta_1 + I_2\theta_2) = \theta_2 - \frac{\omega^2}{2 \times 10^6}(2 \times 1 + 4 \times \theta_2)$$

$T_1 = \omega^2 I_1 \theta_1 = \omega^2 \times 2$

$T_2 = T_1 + \omega^2 I_2 \theta_2 = \omega^2 \times 2 + 4\omega^2 \theta_2$

$T_3 = T_2 + \omega^2 I_3 \theta_3 = \omega^2 \times 2 + 4\omega^2 \theta_2 + 2\omega^2 \theta_3$

These should ideally be evaluated for all values of ω and T_3 plotted against ω. The result is:

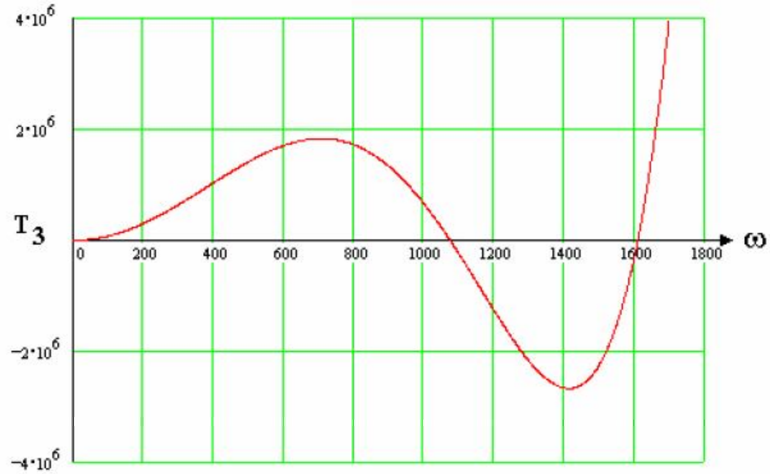

Figure 4

The points where $T_3 = 0$ give the natural frequencies and these are about 1090 and 1610 rad/s.
In an examination environment, plotting this graph is not a practical option. We must start by evaluating in large steps of ω and narrowing it down to the points where T_3 change from plus to minus. This can be very tricky as it is quite possible to miss the critical points if the negative area is a small one.

	ω	θ_1	θ_2	θ_3	T_1	T_2	T_3
1	1	1	1	1	2	6	8
2	10	1	0.9999	0.9996	200	600	800
3	100	1	0.9933	0.9635	$2\text{x}10^4$	$5.97\text{x}10^4$	$7.9\text{x}10^4$
4	1000	1	0.3333	-1.333	$2\text{x}10^6$	$3.33\text{x}10^6$	$0.6667\text{x}10^6$
5	1500	1	-0.5	-0.5	$4.5\text{x}10^6$	0	$-2.25\text{x}10^6$
T3 has gone negative so we need to back.							
6	1250	1	-.0417	-1.474	$3.125\text{x}10^6$	$2.86\text{x}10^6$	$-1.7415\ \text{x}10^6$
7	1100	1	0.1933	-1.484	$2.42\text{x}10^6$	$3.36\text{x}10^6$	$-0.237\ \text{x}10^6$
8	1050	1	0.265	-1.422	$2.2\text{x}10^6$	$3.37\text{x}10^6$	$+0.238\ \text{x}10^6$
9	1070	1	0.2367	-1.45	$2.29\text{x}10^6$	$3.37\text{x}10^6$	$+0.053\ \text{x}10^6$
10	1080	1	0.224	-1.46	$2.33\text{x}10^6$	$3.37\text{x}10^6$	$-0.042\ \text{x}10^6$
11	1075	1	0.23	-1.46	$2.31\text{x}10^6$	$3.37\text{x}10^6$	$+0.0058\ \text{x}10^6$
12	1076	1	0.228	-1.46	$2.31\text{x}10^6$	$3.37\text{x}10^6$	$-0.0037\ \text{x}10^6$
Continuing we find the next point at 1610							
	1610	1	-0.728	+0.454	$5.18\text{x}10^6$	$-2.36\text{x}10^6$	$-0.009\ \text{x}10^6$

The first natural frequency is 1076 rad/s. We would have to carry on finding the next natural frequency is 1610 rad/s. Examiners often give a clue about the natural frequency and this helps to narrow it down but is is a laborious process to carry out in the time allocated.

WORKED EXAMPLE No.3

For the same problem (W.E.2) determine the approximate nodal points.

SOLUTION

This involves plotting the θ values at the rotor.

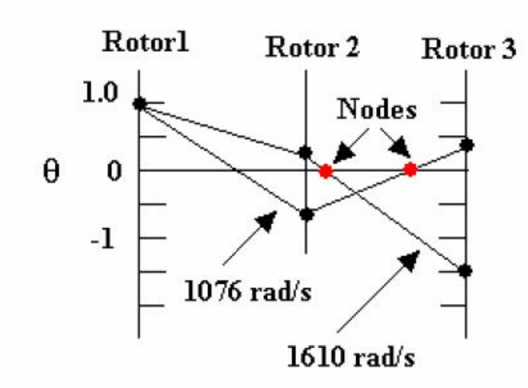

Figure 5

At 1610 rad/s the node between rotor 2 and 3 and close to rotor 2. At 1076 rad/s the node is between rotors 2 and 3 and closer to 3 than 2.

SELF ASSESSMENT EXERCISE No.1

1. A hydraulic motor shaft is supported at the free end in bearings and carries a set of pulley wheels on it. The motor has a moment of inertia of 0.8 kg m^2 and the pulley wheels have a moment of inertia of 2 kg m^2. The shaft has a stiffness of 500 Nm/rad. Calculate the natural frequency of torsional vibrations. (298 Hz)

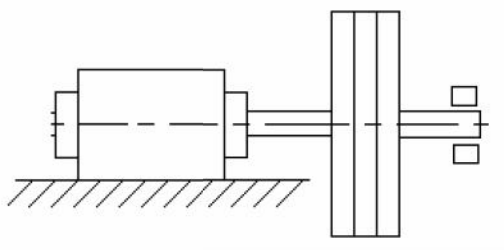

Figure 6

2. A winding motor for raising a lift has the winding wheels mounted on bearings as shown. It is connected with a coupling.

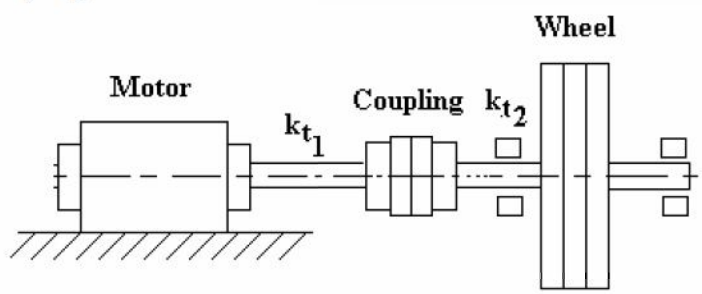

Figure 7

k_{t1} =80 kN m/rad k_{t2} =60 kN m/rad I_{MOTOR} = 2 kg m^2 $I_{COUPLING}$ = 0.8 kg m^2 I_{WHEEL} = 3 kg m^2

Show that there is a natural frequencies of vibration occur between 100 and 200 rad/s and between 400 and 500 rad/s.

3. A gas turbine lay out shown below. There are five moments of inertia and four shaft sections. The data is shown below. Show that there is a natural frequencies close to 300 rad/s and another between 500 and 600 rad/s. Determine in which sections the nodal points occur at each frequency.

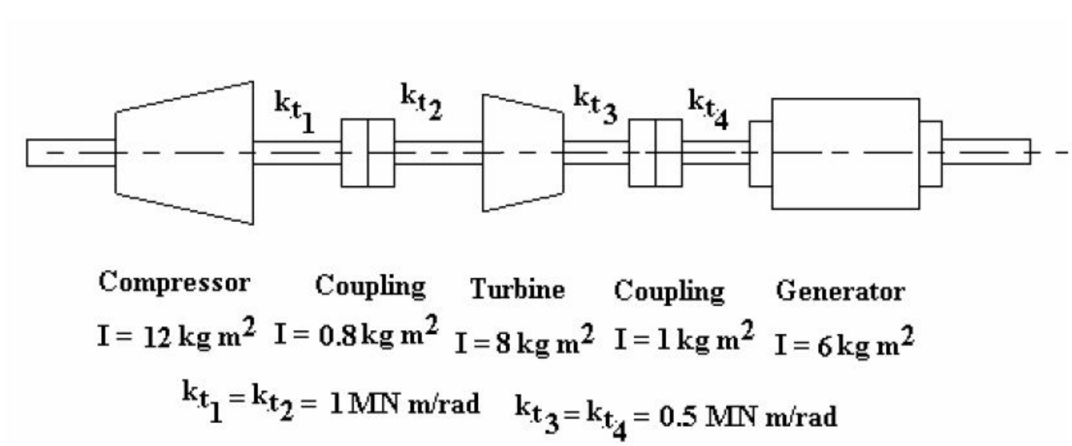

Figure 8